Section 1.3—Rates of Change

Many practical relationships involve interdependent quantities. For example, the volume of a balloon varies with its height above the ground, air temperature varies with elevation, and the surface area of a sphere varies with the length of the radius.

These and other relationships can be described by means of a function, often of the form $y = f(x)$. The **dependent variable**, y, can represent quantities such as volume, air temperature, and area. The **independent variable**, x, can represent quantities such as height, elevation, and length.

We are often interested in how rapidly the dependent variable changes when there is a change in the independent variable. Recall that this concept is called **rate of change**. In this section, we show that an instantaneous rate of change can be calculated by finding the limit of a difference quotient in the same way that we find the slope of a tangent.

Velocity as a Rate of Change

An object moving in a straight line is an example of a rate-of-change model. It is customary to use either a horizontal or vertical line with a specified origin to represent the line of motion. On such a line, movement to the right or upward is considered to be in the positive direction, and movement to the left (or down) is considered to be in the negative direction. An example of an object moving along a line would be a vehicle entering a highway and travelling north 340 km in 4 h.

The average velocity would be $\frac{340}{4} = 85$ km/h, since

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

If $s(t)$ gives the position of the vehicle on a straight section of the highway at time t, then the average rate of change in the position of the vehicle over a time interval is average velocity $= \frac{\Delta s}{\Delta t}$.

INVESTIGATION

You are driving with a broken speedometer on a highway. At any instant, you do not know how fast the car is going. Your odometer readings are given

***t* (h)**	0	1	2	2.5	3
***s*(*t*) (km)**	62	133	210	250	293

A. Determine the average velocity of the car over each interval.
B. The speed limit is 80 km/h. Do any of your results in part A suggest that you were speeding at any time? If so, when?
C. Explain why there may be other times when you were travelling above the posted speed limit.
D. Compute your average velocity over the interval $4 \le t \le 7$, if $s(4) = 375$ km and $s(7) = 609$ km.
E. After 3 h of driving, you decide to continue driving from Goderich to Huntsville, a distance of 345 km. Using the average velocity from part D, how long would it take you to make this trip?

EXAMPLE 1 Reasoning about average velocity

A pebble is dropped from a cliff, 80 m high. After t seconds, the pebble is s metres above the ground, where $s(t) = 80 - 5t^2$, $0 \le t \le 4$.

a. Calculate the average velocity of the pebble between the times $t = 1$ s and $t = 3$ s.
b. Calculate the average velocity of the pebble between the times $t = 1$ s and $t = 1.5$ s.
c. Explain why your answers for parts a and b are different.

Solution

a. $$\text{average velocity} = \frac{\Delta s}{\Delta t}$$
$$s(1) = 75$$
$$s(3) = 35$$
$$\begin{aligned}\text{average velocity} &= \frac{s(3) - s(1)}{3 - 1}\\ &= \frac{35 - 75}{2}\\ &= \frac{-40}{2}\\ &= -20 \text{ m/s}\end{aligned}$$

The average velocity in this 2 s interval is -20 m/s.

b. $$\begin{aligned}s(1.5) &= 80 - 5(1.5)^2\\ &= 68.75\end{aligned}$$
$$\begin{aligned}\text{average velocity} &= \frac{s(1.5) - s(1)}{1.5 - 1}\\ &= \frac{68.75 - 75}{0.5}\\ &= -12.5 \text{ m/s}\end{aligned}$$

The average velocity in this 0.5 s interval is -12.5 m/s.

c. Since gravity causes the velocity to increase with time, the smaller interval of 0.5 s gives a lower average velocity, as well as giving a value closer to the actual velocity at time $t = 1$.

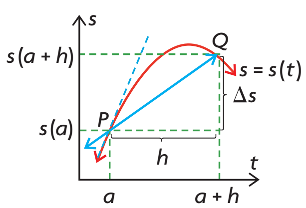
The following table shows the results of similar calculations of the average velocity over successively smaller time intervals:

Time Interval	Average Velocity (m/s)
$1 \le t \le 1.1$	-10.5
$1 \le t \le 1.01$	-10.05
$1 \le t \le 1.001$	-10.005

It appears that, as we shorten the time interval, the average velocity is approaching the value -10 m/s. The average velocity over the time interval $1 \le t \le 1 + h$ is

$$\begin{aligned}
\text{average velocity} &= \frac{s(1 + h) - s(1)}{h} \\
&= \frac{[80 - 5(1 + h)^2] - [80 - 5(1)^2]}{h} \\
&= \frac{75 - 10h - 5h^2 - 75}{h} \\
&= \frac{-10h - 5h^2}{h} \\
&= -10 - 5h, h \ne 0
\end{aligned}$$

If the time interval is very short, then h is small, so $5h$ is close to 0 and the average velocity is close to -10 m/s. The instantaneous velocity when $t = 1$ is defined to be the limiting value of these average values as h approaches 0. Therefore, the velocity (the word "instantaneous" is usually omitted) at time $t = 1$ s is $v = \lim_{h \to 0}(-10 - 5h) = -10$ m/s.



In general, suppose that the position of an object at time t is given by the function $s(t)$. In the time interval from $t = a$ to $t = a + h$, the change in position is $\Delta s = s(a + h) - s(a)$.

The average velocity over this time interval is $\frac{\Delta s}{\Delta t} = \frac{s(a + h) - s(a)}{h}$, which is the same as the slope of the secant PQ where $P(a, s(a))$ and $Q(a + h, s(a + h))$. The **velocity** at a particular time $t = a$ is calculated by finding the limiting value of the average velocity as $h \to 0$.

Instantaneous Velocity

The velocity of an object with position function $s(t)$, at time $t = a$, is

$$v(a) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}$$

Note that the velocity $v(a)$ is the slope of the tangent to the graph of $s(t)$ at $P(a, s(a))$. The speed of an object is the absolute value of its velocity. It indicates how fast an object is moving, whereas velocity indicates both speed and direction (relative to a given coordinate system).

EXAMPLE 2 Selecting a strategy to calculate velocity

A toy rocket is launched straight up so that its height s, in metres, at time t, in seconds, is given by $s(t) = -5t^2 + 30t + 2$. What is the velocity of the rocket at $t = 4$?

Solution

Since $s(t) = -5t^2 + 30t + 2$,

$$\begin{aligned} s(4 + h) &= -5(4 + h)^2 + 30(4 + h) + 2 \\ &= -80 - 40h - 5h^2 + 120 + 30h + 2 \\ &= -5h^2 - 10h + 42 \\ s(4) &= -5(4)^2 + 30(4) + 2 \\ &= 42 \end{aligned}$$

The velocity at $t = 4$ is

$$v(4) = \lim_{h \to 0} \frac{s(4 + h) - s(4)}{h} \quad \text{(Substitute)}$$

$$= \lim_{h \to 0} \frac{[-10h - 5h^2]}{h} \quad \text{(Factor)}$$

$$= \lim_{h \to 0} \frac{h(-10 - 5h)}{h} \quad \text{(Simplify)}$$

$$= \lim_{h \to 0} (-10 - 5h) \quad \text{(Evaluate)}$$

$$= -10$$

Therefore, the velocity of the rocket is 10 m/s downward at $t = 4$ s.

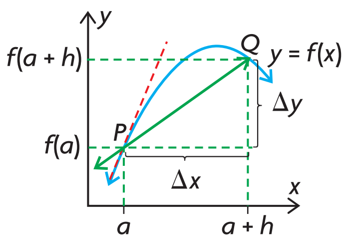
Comparing Average and Instantaneous Rates of Change

Velocity is only one example of the concept of rate of change. In general, suppose that a quantity y depends on x according to the equation $y = f(x)$. As the independent variable changes from a to $a + h$ ($\Delta x = a + h - a = h$), the corresponding change in the dependent variable y is $\Delta y = f(a + h) - f(a)$.

Average Rate of Change

The difference quotient $\frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{h}$ is called the average rate of change in y with respect to x over the interval from $x = a$ to $x = a + h$.

From the diagram, it follows that the average rate of change equals the slope of the secant PQ of the graph of $f(x)$ where $P(a, f(a))$ and $Q(a + h, f(a + h))$. The instantaneous rate of change in y with respect to x when $x = a$ is defined to be the limiting value of the average rate of change as $h \to 0$.



Instantaneous Rates of Change

Therefore, we conclude that the instantaneous rate of change in $y = f(x)$ with respect to x when $x = a$ is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$, provided that the limit exists.

It should be noted that, as with velocity, the instantaneous rate of change in y with respect to x at $x = a$ equals the slope of the tangent to the graph of $y = f(x)$ at $x = a$.

EXAMPLE 3 **Selecting a strategy to calculate instantaneous rate of change**

The total cost, in dollars, of manufacturing x calculators is given by $C(x) = 10\sqrt{x} + 1000$.

a. What is the total cost of manufacturing 100 calculators?
b. What is the rate of change in the total cost with respect to the number of calculators, x, being produced when $x = 100$?

Solution

a. $C(100) = 10\sqrt{100} + 1000 = 1100$

Therefore, the total cost of manufacturing 100 calculators is $1100.

b. The rate of change in the cost at $x = 100$ is given by

$$\lim_{h\to 0}\frac{C(100+h) - C(100)}{h} \qquad \text{(Substitute)}$$

$$= \lim_{h\to 0}\frac{10\sqrt{100+h} + 1000 - 1100}{h}$$

$$= \lim_{h\to 0}\frac{10\sqrt{100+h} - 100}{h} \times \frac{10\sqrt{100+h} + 100}{10\sqrt{100+h} + 100} \qquad \text{(Rationalize the numerator)}$$

$$= \lim_{h\to 0}\frac{100(100+h) - 10\,000}{h(10\sqrt{100+h} + 100)} \qquad \text{(Expand)}$$

$$= \lim_{h\to 0}\frac{100h}{h(10\sqrt{100+h} + 100)} \qquad \text{(Simplify)}$$

$$= \lim_{h\to 0}\frac{100}{(10\sqrt{100+h} + 100)} \qquad \text{(Evaluate)}$$

$$= \frac{100}{(10\sqrt{100+0} + 100)}$$

$$= 0.5$$

Therefore, the rate of change in the total cost with respect to the number of calculators being produced, when 100 calculators are being produced, is $0.50 per calculator.

An Alternative Form for Finding Rates of Change

In Example 1, we determined the velocity of the pebble at $t = 1$ by taking the limit of the average velocity over the interval $1 \le t \le 1 + h$ as h approaches 0.

We can also determine the velocity at $t = 1$ by considering the average velocity over the interval from 1 to a general time t and letting t approach the value 1.

Then, $s(t) = 80 - 5t^2$

$s(1) = 75$

$$\begin{aligned} v(1) &= \lim_{t\to 1} \frac{s(t) - s(1)}{t - 1} \\ &= \lim_{t\to 1} \frac{5 - 5t^2}{t - 1} \\ &= \lim_{t\to 1} \frac{5(1 - t)(1 + t)}{t - 1} \\ &= \lim_{t\to 1} -5(1 + t) \\ &= -10 \end{aligned}$$

In general, the velocity of an object at time $t = a$ is $v(a) = \lim_{t\to a} \frac{s(t) - s(a)}{t - a}$. Similarly, the instantaneous rate of change in $y = f(x)$ with respect to x when $x = a$ is $\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$.

IN SUMMARY

Key Ideas

- The average velocity can be found in the same way that we found the slope of the secant.

  $$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

- The instantaneous velocity is the slope of the tangent to the graph of the position function and is found in the same way that we found the slope of the tangent.

Need to Know

- To find the average velocity (average rate of change) from $t = a$ to $t = a + h$, we can use the difference quotient and the position function $s(t)$

  $$\frac{\Delta s}{\Delta t} = \frac{s(a + h) - s(a)}{h}$$

- The rate of change in the position function, $s(t)$, is the velocity at $t = a$, and we can find it by computing the limiting value of the average velocity as $h \to 0$:

  $$v(a) = \lim_{h\to 0} \frac{s(a + h) - s(a)}{h}$$

Exercise 1.3

PART A

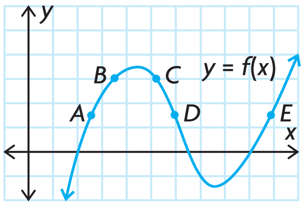
1. The velocity of an object is given by $v(t) = t(t - 4)^2$. At what times, in seconds, is the object at rest?

C 2. Give a geometrical interpretation of the following expressions, if s is a position function:

   a. $\dfrac{s(9) - s(2)$}{7}$ b. $\lim_{h \to 0} \dfrac{s(6 + h) - s(6)}{h}$

3. Give a geometrical interpretation of $\lim_{h \to 0} \dfrac{\sqrt{4 + h} - 2}{h}$.

4. Use the graph to answer each question.



   a. Between which two consecutive points is the average rate of change in the function the greatest?

   b. Is the average rate of change in the function between A and B greater than or less than the instantaneous rate of change at B?

   c. Sketch a tangent to the graph somewhere between points D and E such that the slope of the tangent is the same as the average rate of change in the function between B and C.

5. What is wrong with the statement "The speed of the cheetah was 65 km/h north"?

6. Is there anything wrong with the statement "A school bus had a velocity of 60 km/h for the morning run, which is why it was late arriving"?

PART B

7. A construction worker drops a bolt while working on a high-rise building, 320 m above the ground. After t seconds, the bolt has fallen a distance of s metres, where $s(t) = 320 - 5t^2$, $0 \leq t \leq 8$.

   a. Calculate the average velocity during the first, third, and eighth seconds.

   b. Calculate the average velocity for the interval $3 \leq t \leq 8$.

   c. Calculate the velocity at $t = 2$.

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8. The function $s(t) = 8t(t + 2)$ describes the distance s, in kilometres, that a car has travelled after a time t, in hours, for $0 \leq t \leq 5$.
   a. Calculate the average velocity of the car during the following intervals:
      i. from $t = 3$ to $t = 4$
      ii. from $t = 3$ to $t = 3.1$
      iii. from $t = 3$ to $t = 3.01$
   b. Use your results for part a to approximate the instantaneous velocity of the car at $t = 3$.
   c. Calculate the velocity at $t = 3$.
9. Suppose that a foreign-language student has learned $N(t) = 20t - t^2$ vocabulary terms after t hours of uninterrupted study, where $0 \leq t \leq 10$.
   a. How many terms are learned between time $t = 2$ h and $t = 3$ h?
   b. What is the rate, in terms per hour, at which the student is learning at time $t = 2$ h?

A

10. A medicine is administered to a patient. The amount of medicine M, in milligrams, in 1 mL of the patient's blood, t hours after the injection, is $M(t) = -\frac{1}{3}t^2 + t$, where $0 \leq t \leq 3$.
    a. Find the rate of change in the amount M, 2 h after the injection.
    b. What is the significance of the fact that your answer is negative?
11. The time t, in seconds, taken by an object dropped from a height of s metres to reach the ground is given by the formula $t = \sqrt{\frac{s}{5}}$. Determine the rate of change in time with respect to height when the object is 125 m above the ground.
12. Suppose that the temperature T, in degrees Celsius, varies with the height h, in kilometres, above Earth's surface according to the equation $T(h) = \frac{60}{h + 2}$. Find the rate of change in temperature with respect to height at a height of 3 km.
13. A spaceship approaching touchdown on a distant planet has height h, in metres, at time t, in seconds, given by $h = 25t^2 - 100t + 100$. When does the spaceship land on the surface? With what speed does it land (assuming it descends vertically)?
14. A manufacturer of soccer balls finds that the profit from the sale of x balls per week is given by $P(x) = 160x - x^2$ dollars.
    a. Find the profit on the sale of 40 soccer balls per week.
    b. Find the rate of change in profit at the production level of 40 balls per week.
    c. Using a graphing calculator, graph the profit function and, from the graph, determine for what sales levels of x the rate of change in profit is positive.

15. Use the alternate definition $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ to calculate the instantaneous rate of change of $f(x)$ at the given point or value of x.

a. $f(x) = -x^2 + 2x + 3, (-2, -5)$

b. $f(x) = \frac{x}{x - 1}, x = 2$

c. $f(x) = \sqrt{x + 1}, x = 24$

16. The average annual salary of a professional baseball player can be modelled by the function $S(x) = 246 + 64x - 8.9x^2 + 0.95x^3$, where S represents the average annual salary, in thousands of dollars, and x is the number of years since 1982. Determine the rate at which the average salary was changing in 2005.

17. The motion of an avalanche is described by $s(t) = 3t^2$, where s is the distance, in metres, travelled by the leading edge of the snow at t seconds.

a. Find the distance travelled from 0 s to 5 s.

b. Find the rate at which the avalanche is moving from 0 s to 10 s.

c. Find the rate at which the avalanche is moving at 10 s.

d. How long, to the nearest second, does the leading edge of the snow take to move 600 m?

PART C

T 18. Let (a, b) be any point on the graph of $y = \frac{1}{x}, x \neq 0$. Prove that the area of the triangle formed by the tangent through (a, b) and the coordinate axes is 2.

19. MegaCorp's total weekly cost to produce x pencils can be written as $C(x) = F + V(x)$, where F, a constant, represents fixed costs such as rent and utilities and $V(x)$ represents variable costs, which depend on the production level x. Show that the rate of change in the weekly cost is independent of fixed costs.

20. A circular oil spill on the surface of the ocean spreads outward. Find the approximate rate of change in the area of the oil slick with respect to its radius when the radius is 100 m.

21. Show that the rate of change in the volume of a cube with respect to its edge length is equal to half the surface area of the cube.

22. Determine the instantaneous rate of change in

a. the surface area of a spherical balloon (as it is inflated) at the point in time when the radius reaches 10 cm

b. the volume of a spherical balloon (as it is deflated) at the point in time when the radius reaches 5 cm