Section 1.4—The Limit of a Function

The notation $\lim_{x \to a} f(x) = L$ is read "the limit of $f(x)$ as x approaches a equals L" and means that the value of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a (but not equal to a). But $\lim_{x \to a} f(x)$ exists if and only if the limiting value from the left equals the limiting value from the right. We shall use this definition to evaluate some limits.

Note: This is an intuitive explanation of the limit of a function. A more precise definition using inequalities is important for advanced work but is not necessary for our purposes.

INVESTIGATION 1 Determine the limit of $y = x^2 - 1$, as x approaches 2.

A. Copy and complete the table of values.

x	1	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5	3
$y = x^2 - 1$											

B. As x approaches 2 from the left, starting at $x = 1$, what is the approximate value of y?

C. As x approaches 2 from the right, starting at $x = 3$, what is the approximate value of y?

D. Graph $y = x^2 - 1$ using graphing software or graph paper.

E. Using arrows, illustrate that, as we choose a value of x that is closer and closer to $x = 2$, the value of y gets closer and closer to a value of 3.

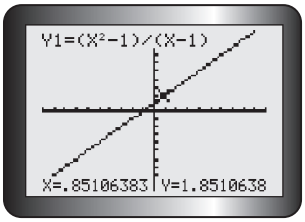
F. Explain why the limit of $y = x^2 - 1$ exists as x approaches 2, and give its approximate value.

EXAMPLE 1 Determine $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ by graphing.

Solution

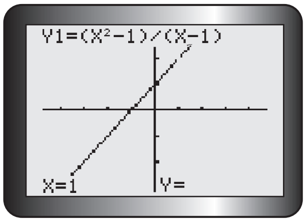
On a graphing calculator, display the graph of $f(x) = \frac{x^2 - 1}{x - 1}, x \neq 1$.

The graph shown on your calculator is a line $(f(x) = x + 1)$, whereas it should be a line with point (1, 2) deleted $(f(x) = x + 1, x \neq 1)$. The WINDOW used is $X_{min} = -10$, $X_{max} = 10$, $X_{scl} = 1$, and similarly for Y. Use the TRACE function to find $X = 0.85106383$, $Y = 1.8510638$ and $X = 1.0638298$, $Y = 2.0638298$.



Click ZOOM; select 4:ZDecimal, ENTER. Now, the graph of $f(x) = \frac{x^2 - 1}{x - 1}$ is displayed as a straight line with point (1, 2) deleted. The WINDOW has new values, too.

Use the TRACE function to find $X = 0.9$, $Y = 1.9$; $X = 1$, Y has no value given; and $X = 1.1$, $Y = 2.1$.



We can estimate $\lim_{x \to 1} f(x)$. As x approaches 1 from the left, written as "$x \to 1^-$", we observe that $f(x)$ approaches the value 2 from below. As x approaches 1 from the right, written as $x \to 1^+$, $f(x)$ approaches the value 2 from above.

We say that the limit at $x = 1$ exists only if the value approached from the left is the same as the value approached from the right. From this investigation, we conclude that $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$.

EXAMPLE 2 Selecting a table of values strategy to evaluate a limit

Determine $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ by using a table.

Solution

We select sequences of numbers for $x \to 1^-$ and $x \to 1^+$.

x approaches 1 from the left $\rightarrow$						$\leftarrow$ x approaches 1 from the right					
x	0	0.5	0.9	0.99	0.999	1	1.001	1.01	1.1	1.5	2
$\frac{x^2 - 1}{x - 1}$	1	1.5	1.9	1.99	1.999	undefined	2.001	2.01	2.1	2.5	3
$f(x) = \frac{x^2 - 1}{x - 1}$ approaches 2 from below $\rightarrow$						$\leftarrow$ $f(x) = \frac{x^2 - 1}{x - 1}$ approaches 2 from above					

This pattern of numbers suggests that $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$, as we found when graphing in Example 1.

EXAMPLE 3 Selecting a graphing strategy to evaluate a limit

Tech Support
For help graphing piecewise functions on a graphing calculator, see Technology Appendix p. 607.

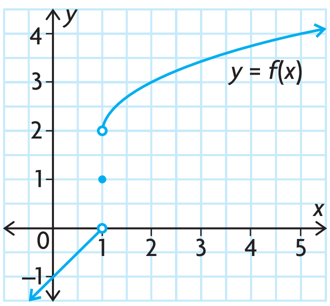
Sketch the graph of the piecewise function:

$$f(x) = \begin{cases} x - 1, \text{ if } x < 1 \\ 1, \text{ if } x = 1 \\ 2 + \sqrt{x - 1}, \text{ if } x > 1 \end{cases}$$

Determine $\lim_{x \to 1} f(x)$.

Solution

The graph of the function f consists of the line $y = x - 1$ for $x < 1$, the point $(1, 1)$ and the square root function $y = 2 + \sqrt{x - 1}$ for $x > 1$. From the graph of $f(x)$, observe that the limit of $f(x)$ as $x \to 1$ depends on whether $x < 1$ or $x > 1$. As $x \to 1^-$, $f(x)$ approaches the value of 0 from below. We write this as $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x - 1) = 0$.



Similarly, as $x \to 1^+$, $f(x)$ approaches the value 2 from above. We write this as $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(2 + \sqrt{x - 1}\right) = 2$. (This is the same when $x = 1$ is substituted into the expression $2 + \sqrt{x - 1}$.) These two limits are referred to as one-sided limits because, in each case, only values of x on one side of $x = 1$ are considered. How-ever, the one-sided limits are unequal—$\lim_{x \to 1^-} f(x) = 0 \neq 2 = \lim_{x \to 1^+} f(x)$—or more briefly, $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$. This implies that $f(x)$ does not approach a single value as $x \to 1$. We say "the limit of $f(x)$ as $x \to 1$ does not exist" and write "$\lim_{x \to 1} f(x)$ does not exist." This may be surprising, since the function $f(x)$ was defined at $x = 1$—that is, $f(1) = 1$. We can now summarize the ideas introduced in these examples.

Limits and Their Existence

We say that the number L is the limit of a function $y = f(x)$ as x approaches the value a, written as $\lim_{x \to a} f(x) = L$, if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$. Otherwise, $\lim_{x \to a} f(x)$ does not exist.

IN SUMMARY

Key Idea

- The limit of a function $y = f(x)$ at $x = a$ is written as $\lim_{x \to a} f(x) = L$, which means that $f(x)$ approaches the value L as x approaches the value a from both the left and right side.

Need to Know

- $\lim_{x \to a} f(x)$ may exist even if $f(a)$ is not defined.
- $\lim_{x \to a} f(x)$ can be equal to $f(a)$. In this case, the graph of $f(x)$ passes through the point $(a, f(a))$.
- If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$, then L is the limit of $f(x)$ as x approaches a, that is $\lim_{x \to a} f(x) = L$.

Exercise 1.4

PART A

1. What do you think is the appropriate limit of each sequence?
   a. 0.7, 0.72, 0.727, 0.7272, . . .
   b. 3, 3.1, 3.14, 3.141, 3.1415, 3.141 59, 3.141 592, . . .

C 2. Explain a process for finding a limit.

3. Write a concise description of the meaning of the following:
   a. a right-sided limit   b. a left-sided limit   c. a (two-sided) limit

4. Calculate each limit.

a. $\lim_{x \to -5} x$ c. $\lim_{x \to 10} x^2$ e. $\lim_{x \to 1} 4$

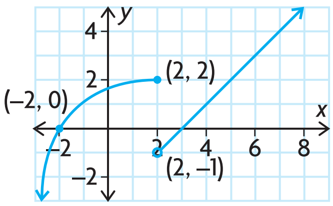
b. $\lim_{x \to 3} (x + 7)$ d. $\lim_{x \to -2} (4 - 3x^2)$ f. $\lim_{x \to 3} 2^x$

5. Determine $\lim_{x \to 4} f(x)$, where $f(x) = \begin{cases} 1, \text{ if } x \neq 4 \\ -1, \text{ if } x = 4 \end{cases}$.

PART B

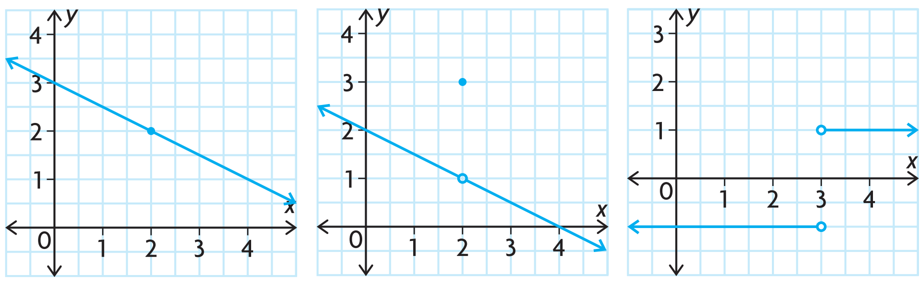
6. For the function $f(x)$ in the graph below, determine the following:

a. $\lim_{x \to -2^+} f(x)$ b. $\lim_{x \to 2^-} f(x)$ c. $\lim_{x \to 2^+} f(x)$ d. $f(2)$



K 7. Use the graph to find the limit, if it exists.

a. $\lim_{x \to 2} f(x)$ b. $\lim_{x \to 2} f(x)$ c. $\lim_{x \to 3} f(x)$



8. Evaluate each limit.

a. $\lim_{x \to -1} (9 - x^2)$ b. $\lim_{x \to 0} \sqrt{\dfrac{x + 20}{2x + 5}}$ c. $\lim_{x \to 5} \sqrt{x - 1}$

9. Find $\lim_{x \to 2} (x^2 + 1)$, and illustrate your result with a graph indicating the limiting value.

10. Evaluate each limit. If the limit does not exist, explain why.

a. $\lim_{x \to 0^+} x^4$ c. $\lim_{x \to 3^-} (x^2 - 4)$ e. $\lim_{x \to 3^+} \dfrac{1}{x + 2}$

b. $\lim_{x \to 2^-} (x^2 - 4)$ d. $\lim_{x \to 1^+} \dfrac{1}{x - 3}$ f. $\lim_{x \to 3} \dfrac{1}{x - 3}$

11. For each function, sketch the graph of the function. Determine the indicated limit if it exists.

a. $f(x) = \begin{cases} x + 2, \text{ if } x < -1 \\ -x + 2, \text{ if } x \geq -1 \end{cases}; \lim_{x \to -1} f(x)$

b. $f(x) = \begin{cases} -x + 4, \text{ if } x \leq 2 \\ -2x + 6, \text{ if } x > 2 \end{cases}; \lim_{x \to 2} f(x)$

c. $f(x) = \begin{cases} 4x, \text{ if } x \geq \frac{1}{2} \\ \frac{1}{x}, \text{ if } x < \frac{1}{2} \end{cases}; \lim_{x \to \frac{1}{2}} f(x)$

d. $f(x) = \begin{cases} 1, \text{ if } x < -0.5 \\ x^2 - 0.25, \text{ if } x \geq -0.5 \end{cases}; \lim_{x \to -0.5} f(x)$

A 12. Sketch the graph of any function that satisfies the given conditions.

a. $f(1) = 1, \lim_{x \to 1^+} f(x) = 3, \lim_{x \to 1^-} f(x) = 2$

b. $f(2) = 1, \lim_{x \to 2} f(x) = 0$

c. $f(x) = 1$, if $x < 1$ and $\lim_{x \to 1^+} f(x) = 2$

d. $f(3) = 0, \lim_{x \to 3^+} f(x) = 0$

13. Let $f(x) = mx + b$, where m and b are constants. If $\lim_{x \to 1} f(x) = -2$ and $\lim_{x \to -1} f(x) = 4$, find m and b.

PART C

T 14. Determine the real values of a, b, and c for the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, that satisfy the conditions $f(0) = 0$, $\lim_{x \to 1} f(x) = 5$, and $\lim_{x \to -2} f(x) = 8$.

15. The fish population, in thousands, in a lake at time t, in years, is modelled by the following function:

$$p(t) = \begin{cases} 3 + \frac{1}{12}t^2, \text{ if } 0 \leq t \leq 6 \\ 2 + \frac{1}{18}t^2, \text{ if } 6 < t \leq 12 \end{cases}$$

This function describes a sudden change in the population at time $t = 6$, due to a chemical spill.

a. Sketch the graph of $p(t)$.

b. Evaluate $\lim_{t \to 6^-} p(t)$ and $\lim_{t \to 6^+} p(t)$.

c. Determine how many fish were killed by the spill.

d. At what time did the population recover to the level before the spill?