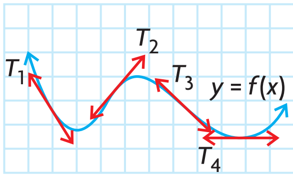
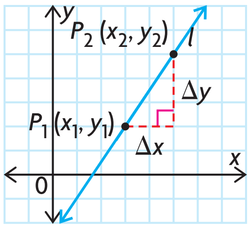
Section 1.2—The Slope of a Tangent

You are familiar with the concept of a **tangent** to a curve. What geometric interpretation can be given to a tangent to the graph of a function at a point P? A tangent is the straight line that most resembles the graph near a point. Its slope tells how steep the graph is at the point of tangency. In the figure below, four tangents have been drawn.



The goal of this section is to develop a method for determining the slope of a tangent at a given point on a curve. We begin with a brief review of lines and slopes.

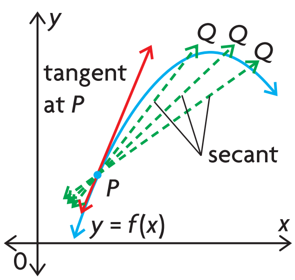
Lines and Slopes



The slope m of the line joining points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is defined as $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

The equation of the line l in point-slope form is $\frac{y - y_1}{x - x_1} = m$ or $y - y_1 = m(x - x_1)$. The equation in slope–y-intercept form is $y = mx + b$, where b is the y-intercept of the line.

To determine the equation of a tangent to a curve at a given point, we first need to know the slope of the tangent. What can we do when we only have one point? We proceed as follows:



Consider a curve $y = f(x)$ and a point P that lies on the curve. Now consider another point Q on the curve. The line joining P and Q is called a **secant**. Think of Q as a moving point that slides along the curve toward P, so that the slope of the secant PQ becomes a progressively better estimate of the slope of the tangent at P.

This suggests the following definition of the slope of the tangent:

Slope of a Tangent

The slope of the tangent to a curve at a point P is the limiting slope of the secant PQ as the point Q slides along the curve toward P. In other words, the slope of the tangent is said to be the **limit** of the slope of the secant as Q approaches P along the curve.

We will illustrate this idea by finding the slope of the tangent to the parabola $y = x^2$ at $P(3, 9)$.

INVESTIGATION 1

A. Determine the y-coordinates of the following points that lie on the graph of the parabola $y = x^2$:

i) $Q_1(3.5, y)$ ii) $Q_2(3.1, y)$ iii) $Q_3(3.01, y)$ iv) $Q_4(3.001, y)$

B. Calculate the slopes of the secants through $P(3, 9)$ and each of the points Q_1, Q_2, Q_3, and Q_4.

C. Determine the y-coordinates of each point on the parabola, and then repeat part B using the following points.

i) $Q_5(2.5, y)$ ii) $Q_6(2.9, y)$ iii) $Q_7(2.99, y)$ iv) $Q_8(2.999, y)$

D. Use your results from parts B and C to estimate the slope of the tangent at point $P(3, 9)$.

E. Graph $y = x^2$ and the tangent to the graph at $P(3, 9)$.

In this investigation, you found the slope of the tangent by finding the limiting value of the slopes of a sequence of secants. Since we are interested in points Q that are close to $P(3, 9)$ on the parabola $y = x^2$ it is convenient to write Q as $(3 + h, (3 + h)^2)$, where h is a very small nonzero number. The variable h determines the position of Q on the parabola. As Q slides along the parabola toward P, h will take on values successively smaller and closer to zero. We say that "h approaches zero" and use the notation "$h \to 0$."

INVESTIGATION 2

A. Using technology or graph paper, draw the parabola $f(x) = x^2$.

B. Let P be the point $(1, 1)$.

C. Determine the slope of the secant through Q_1 and $P(1, 1)$, Q_2 and $P(1, 1)$ and so on, for points $Q_1(1.5, f(1.5))$, $Q_2(1.1, f(1.1))$, $Q_3(1.01, f(1.01))$, $Q_4(1.001, f(1.001))$, and $Q_5(1.0001, f(1.0001))$.

D. Draw these secants on the same graph you created in part A.

E. Use your results to estimate the slope of the tangent to the graph of f at point P.

F. Draw the tangent at point $P(1, 1)$.

INVESTIGATION 3

A. Determine an expression for the slope of the secant PQ through points $P(3, 9)$ and $Q(3 + h, (3 + h)^2)$.

B. Explain how you could use the expression in a part A to predict the slope of the tangent to the parabola $f(x) = x^2$ at point $P(3, 9)$.

The slope of the tangent to the parabola at point P is the limiting slope of the secant line PQ as point Q slides along the parabola; that is, as $h \to 0$, we write "$\lim_{h \to 0}$" as the abbreviation for "limiting value as h approaches 0."

Therefore, from the investigation, the slope of the tangent at a point P is

$$\lim_{h \to 0} (\text{slope of the secant } PQ).$$

EXAMPLE 1

Reasoning about the slope of a tangent as a limiting value

Determine the slope of the tangent to the graph of the parabola $f(x) = x^2$ at $P(3, 9)$.

Solution

Using points $P(3, 9)$ and $Q(3 + h, (3 + h)^2)$, $h \neq 0$, the slope of the secant PQ is

$$\frac{\Delta x}{\Delta y} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{(Substitute)}$$

$$= \frac{(3 + h)^2 - 9}{3 + h - 3} \quad \text{(Expand)}$$

$$= \frac{9 + 6h + h^2 - 9}{h} \quad \text{(Simplify and factor)}$$

$$= \frac{h(6 + h)}{h} \quad \text{(Divide by the common factor of } h\text{)}$$

$$= (6 + h)$$

As $h \to 0$, the value of $(6 + h)$ approaches 6, and thus $\lim_{h \to 0}(6 + h) = 6$.

We conclude that the slope of the tangent at $P(3, 9)$ to the parabola $y = x^2$ is 6.

EXAMPLE 2 Selecting a strategy involving a series of secants to estimate the slope of a tangent

Tech | Support

For help graphing functions using a graphing calculator, see Technology Appendix p. 597.

a. Use your calculator to graph the parabola $y = -\frac{1}{8}(x + 1)(x - 7)$. Plot the points on the parabola from $x = -1$ to $x = 6$, where x is an integer.
b. Determine the slope of the secants using each point from part a and point $P(5, 1.5)$.
c. Use the result of part b to estimate the slope of the tangent at $P(5, 1.5)$.

Solution

a. Using the x-intercepts of -1 and 7, the equation of the axis of symmetry is $x = \frac{-1 + 7}{2} = 3$, so the x-coordinate of the vertex is 3.

Substitute $x = 3$ into $y = -\frac{1}{8}(x + 1)(x - 7)$.

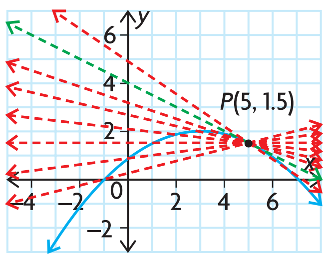
$y = -\frac{1}{8}(3 + 1)(3 - 7) = 2$

Therefore, the vertex is $(3, 2)$.

The y-intercept of the parabola is $\frac{7}{8}$.

The points on the parabola are $(-1, 0)$, $(0, 0.875)$, $(1, 1.5)$, $(2, 1.875)$, $(3, 2)$, $(4, 1.875)$, $(5, 1.5)$, and $(6, 0.875)$.

The parabola and the secants through each point and point $P(5, 1.5)$ are shown in red. The tangent through $P(5, 1.5)$ is shown in green.

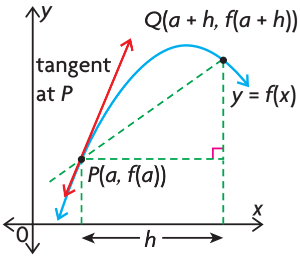


b. Using points $(-1, 0)$ and $P(5, 1.5)$, the slope is $m = \frac{1.5 - 0}{5 - (-1)} = 0.25$. Using the other points and $P(5, 1.5)$, the slopes are 0.125, 0, -0.125, -0.25, -0.375, and -0.625, respectively.

c. The slope of the tangent at $P(5, 1.5)$ is between -0.375 and -0.625. It can be determined to be -0.5 using points closer and closer to $P(5, 1.5)$.

The Slope of a Tangent at an Arbitrary Point

We can now generalize the method used above to derive a formula for the slope of the tangent to the graph of any function $y = f(x)$.



Let $P(a, f(a))$ be a fixed point on the graph of $y = f(x)$, and let $Q(x, y) = Q(x, f(x))$ represent any other point on the graph. If Q is a horizontal distance of h units from P, then $x = a + h$ and $y = f(a + h)$. Point Q then has coordinates $Q(a + h, f(a + h))$.

The slope of the secant PQ is $\frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}$.

This quotient is fundamental to calculus and is referred to as the **difference quotient**. Therefore, the slope m of the tangent at $P(a, f(a))$ is $\lim_{h \to 0}$ (slope of the secant PQ), which may be written as $m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.

Slope of a Tangent as a Limit

The slope of the tangent to the graph $y = f(x)$ at point $P(a, f(a))$ is

$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$, if this limit exists.

EXAMPLE 3 **Connecting limits and the difference quotient to the slope of a tangent**

a. Using the definition of the slope of a tangent, determine the slope of the tangent to the curve $y = -x^2 + 4x + 1$ at the point determined by $x = 3$.
b. Determine the equation of the tangent.
c. Sketch the graph of $y = -x^2 + 4x + 1$ and the tangent at $x = 3$.

Solution

a. The slope of the tangent can be determined using the expression above. In this example, $f(x) = -x^2 + 4x + 1$ and $a = 3$.

Then $f(3) = -(3)^2 + 4(3) + 1 = 4$

and $f(3 + h) = -(3 + h)^2 + 4(3 + h) + 1$

$= -9 - 6h - h^2 + 12 + 4h + 1$

$= -h^2 - 2h + 4$

The slope of the tangent at $(3, 4)$ is

$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$ (Substitute)

$$= \lim_{h \to 0} \frac{[-h^2 - 2h + 4] - 4}{h}$$ (Simplify and factor)

$$= \lim_{h \to 0} \frac{h(-h - 2)}{h}$$ (Divide by the common factor)

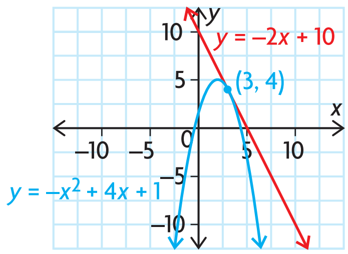
$$= \lim_{h \to 0} (-h - 2)$$ (Evaluate)

$$= -2$$

The slope of the tangent at $x = 3$ is -2.

b. The equation of the tangent at $(3, 4)$ is $\frac{y - 4}{x - 3} = -2$, or $y = -2x + 10$.

c. Using graphing software, we obtain



EXAMPLE 4 Selecting a limit strategy to determine the slope of a tangent

Determine the slope of the tangent to the rational function $f(x) = \frac{3x + 6}{x}$ at point $(2, 6)$.

Solution

Using the definition, the slope of the tangent at $(2, 6)$ is

$$m = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$$ (Substitute)

$$= \lim_{h \to 0} \frac{\frac{6 + 3h + 6}{2 + h} - 6}{h}$$ (Determine a common denominator)

$$= \lim_{h \to 0} \frac{\frac{6 + 3h + 6}{2 + h} - \frac{6(2 + h)}{2 + h}}{h}$$ (Simplify)

$$= \lim_{h \to 0} \frac{\frac{12 + 3h - 12 - 6h}{2 + h}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-3h}{2 + h}}{h}$$ (Multiply by the reciprocal)

$$= \lim_{h \to 0} \frac{-3h}{2 + h} \times \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-3}{2 + h}$$ (Evaluate)

$$= -1.5$$

Therefore, the slope of the tangent to $f(x) = \frac{3x + 6}{x}$ at $(2, 6)$ is -1.5.

EXAMPLE 5 Determining the slope of a line tangent to a root function

Find the slope of the tangent to $f(x) = \sqrt{x}$ at $x = 9$.

Solution

$$f(9) = \sqrt{9} = 3$$

$$f(9 + h) = \sqrt{9 + h}$$

Using the limit of the difference quotient, the slope of the tangent at $x = 9$ is

$$m = \lim_{h \to 0} \frac{f(9 + h) - f(9)}{h}$$ (Substitute)

$$= \lim_{h \to 0} \frac{\sqrt{9 + h} - 3}{h}$$ (Rationalize the numerator)

$$= \lim_{h \to 0} \frac{\sqrt{9 + h} - 3}{h} \times \frac{\sqrt{9 + h} + 3}{\sqrt{9 + h} + 3}$$

$$= \lim_{h \to 0} \frac{(9 + h) - 9}{h(\sqrt{9 + h} + 3)}$$ (Simplify)

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{9 + h} + 3)}$$ (Divide by the common factor of *h*)

$$= \lim_{h \to 0} \frac{1}{\sqrt{9 + h} + 3}$$ (Evalute)

$$= \frac{1}{6}$$

Therefore, the slope of the tangent to $f(x) = \sqrt{x}$ at $x = 9$ is $\frac{1}{6}$.

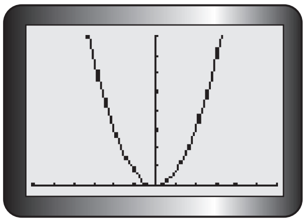
INVESTIGATION 4

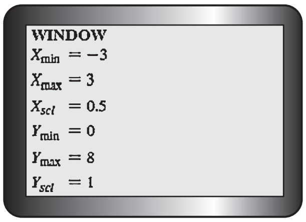
Tech | *Support*

For help graphing functions, tracing, and using the table feature on a graphing calculator, see Technology Appendices p. 597 and p. 599.

A graphing calculator can help us estimate the slope of a tangent at a point. The exact value can then be found using the definition of the slope of the tangent using the difference quotient. For example, suppose that we wish to find the slope of the tangent to $y = f(x) = x^3$ at $x = 1$.

A. Graph $Y_1 = \frac{((x + 0.01)^3 - x^3)}{0.01}$.





B. Explain why the values for the WINDOW were chosen.
Observe that the function entered in Y_1 is the difference quotient $\frac{f(a + h) - f(a)}{h}$ for $f(x) = x^3$ and $h = 0.01$. Remember that this approximates the slope of the tangent and not the graph of $f(x) = x^3$.

C. Use the TRACE function to find $X = 1.0212766$, $Y = 3.159756$.
This means that the slope of the secant passing through the points where $x = 1$ and $x = 1 + 0.01 = 1.01$ is about 3.2. The value 3.2 could be used as an approximation for the slope of the tangent at $x = 1$.

D. Can you improve this approximation? Explain how you could improve your estimate. Also, if you use different WINDOW values, you can see a different-sized, or differently centred, graph.

E. Try once again by setting $X_{min} = -9$, $X_{max} = 10$, and note the different appearance of the graph. Use the TRACE function to find $X = 0.90425532$, $Y = 2.4802607$, and then $X = 1.106383$, $Y = 3.7055414$. What is your guess for the slope of the tangent at $x = 1$ now? Explain why only estimation is possible.

F. Another way of using a graphing calculator to approximate the slope of the tangent is to consider h as the variable in the difference quotient. For this example, $f(x) = x^3$ at $x = 1$, look at $\frac{f(a + h) - f(a)}{h} = \frac{(1 + h)^3 - 1^3}{h}$.

G. Trace values of h as $h \to 0$. You can use the table or graph function of your calculator. Graphically, we say that we are looking at $\frac{(1 + h)^3 - 1}{h}$ in the neighbourhood of $h = 0$. To do this, graph $y = \frac{(1 + x)^3 - 1}{x}$ and examine the value of the function as $x \to 0$.

IN SUMMARY

Key Ideas

- The slope of the tangent to a curve at a point P is the limit of the slopes of the secants PQ as Q moves closer to P.

  $m_{\text{tangent}} = \lim_{Q \to P} (\textit{slope of secant } PQ)$
- The slope of the tangent to the graph of $y = f(x)$ at $P(a, f(a))$ is given by

  $m_{\text{tangent}} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.

Need to Know

- To find the slope of the tangent at a point $P(a, f(a))$,
  - find the value of $f(a)$
  - find the value of $f(a + h)$
  - evaluate $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$

Exercise 1.2

PART A

1. Calculate the slope of the line through each pair of points.

   a. $(2, 7), (-3, -8)$

   b. $\left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, -\frac{7}{2}\right)$

   c. $(6.3, -2.6), (1.5, -1)$

2. Determine the slope of a line perpendicular to each of the following:

   a. $y = 3x - 5$

   b. $13x - 7y - 11 = 0$

3. State the equation and sketch the graph of each line described below.

   a. passing through $(-4, -4)$ and $\left(\frac{5}{3}, -\frac{5}{3}\right)$

   b. having slope 8 and y-intercept 6

   c. having x-intercept 5 and y-intercept -3

   d. passing through $(5, 6)$ and $(5, -9)$

4. Simplify each of the following difference quotients:

a. $\frac{(5 + h)^3 - 125}{h}$

b. $\frac{(3 + h)^4 - 81}{h}$

c. $\frac{\frac{1}{1 + h} - 1}{h}$

d. $\frac{3(1 + h)^2 - 3}{h}$

e. $\frac{\frac{3}{4 + h} - \frac{3}{4}}{h}$

f. $\frac{\frac{-1}{2 + h} + \frac{1}{2}}{h}$

5. Rationalize the numerator of each expression to obtain an equivalent expression.

a. $\frac{\sqrt{16 + h} - 4}{h}$

b. $\frac{\sqrt{h^2 + 5h + 4} - 2}{h}$

c. $\frac{\sqrt{5 + h} - \sqrt{5}}{h}$

PART B

6. Determine an expression, in simplified form, for the slope of the secant PQ.

a. $P(1, 3)$, $Q(1 + h, f(1 + h))$, where $f(x) = 3x^2$

b. $P(1, 3)$, $Q(1 + h, (1 + h)^3 + 2)$

c. $P(9, 3)$, $Q(9 + h, \sqrt{9 + h})$

K

7. Consider the function $f(x) = x^3$.

a. Copy and complete the following table of values. P and Q are points on the graph of $f(x)$.

P	***Q***	**Slope of Line *PQ***
(2,)	(3,)	
(2,)	(2.5,)	
(2,)	(2.1,)	
(2,)	(2.01,)	
(2,)	(1,)	
(2,)	(1.5,)	
(2,)	(1.9,)	
(2,)	(1.99,)	

b. Use your results for part a to approximate the slope of the tangent to the graph of $f(x)$ at point P.

c. Calculate the slope of the secant PQ, where the x-coordinate of Q is $2 + h$.

d. Use your result for part c to calculate the slope of the tangent to the graph of $f(x)$ at point P.

e. Compare your answers for parts b and d.

f. Sketch the graph of $f(x)$ and the tangent to the graph at point P.

8. Determine the slope of the tangent to each curve at the given value of x.

a. $y = 3x^2, x = -2$ b. $y = x^2 - x, x = 3$ c. $y = x^3, x = -2$

9. Determine the slope of the tangent to each curve at the given value of x.

a. $y = \sqrt{x - 2}, x = 3$

b. $y = \sqrt{x - 5}, x = 9$

c. $y = \sqrt{5x - 1}, x = 2$

10. Determine the slope of the tangent to each curve at the given value of x.

a. $y = \frac{8}{x}, x = 2$ b. $y = \frac{8}{3 + x}, x = 1$ c. $y = \frac{1}{x + 2}, x = 3$

11. Determine the slope of the tangent to each curve at the given point.

a. $y = x^2 - 3x, (2, -2)$

b. $f(x) = \frac{4}{x}, (-2, -2)$

c. $y = 3x^3, (1, 3)$

d. $y = \sqrt{x - 7}, (16, 3)$

e. $y = \sqrt{25 - x^2}, (3, 4)$

f. $y = \frac{4 + x}{x - 2}, (8, 2)$

12. Sketch the graph of the function in question 11, part e. Show that the slope of the tangent can be found using the properties of circles.

C 13. Explain how you would approximate the slope of the tangent at a point without using the definition of the slope of the tangent.

14. Using technology, sketch the graph of $y = \frac{3}{4}\sqrt{16 - x^2}$. Explain how the slope of the tangent at $P(0, 3)$ can be found without using the difference quotient.

15. Determine the equation of the tangent to $y = x^2 - 3x + 1$ at $(3, 1)$.

16. Determine the equation of the tangent to $y = x^2 - 7x + 12$ where $x = 2$.

17. For $f(x) = x^2 - 4x + 1$, find

a. the coordinates of point A, where $x = 3$,

b. the coordinates of point B, where $x = 5$

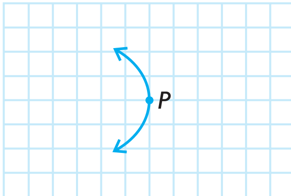
c. the equation of the secant AB

d. the equation of the tangent at A

e. the equation of the tangent at B

18. Copy the following figures. Draw an approximate tangent for each curve at point P and estimate its slope.

a.



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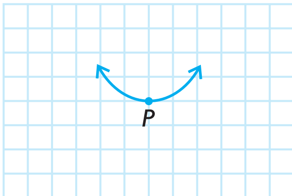

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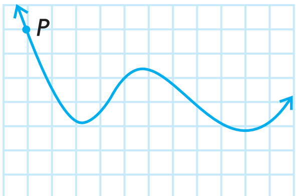

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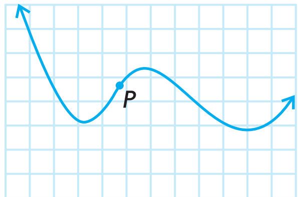

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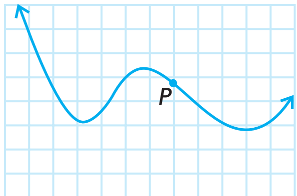

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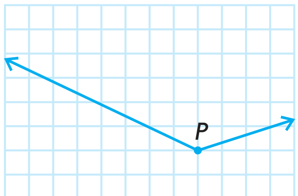

19. Find the slope of the demand curve $D(p) = \frac{20}{\sqrt{p - 1}}$, $p > 1$, at point $(5, 10)$.

A 20. It is projected that, t years from now, the circulation of a local newspaper will be $C(t) = 100t^2 + 400t + 5000$. Find how fast the circulation is increasing after 6 months. *Hint*: Find the slope of the tangent when $t = 0.5$.

T 21. Find the coordinates of the point on the curve $f(x) = 3x^2 - 4x$ where the tangent is parallel to the line $y = 8x$.

22. Find the points on the graph of $y = \frac{1}{3}x^3 - 5x - \frac{4}{x}$ at which the tangent is horizontal.

PART C

23. Show that, at the points of intersection of the quadratic functions $y = x^2$ and $y = \frac{1}{2} - x^2$, the tangents to the functions are perpendicular.

24. Determine the equation of the line that passes through $(2, 2)$ and is parallel to the line tangent to $y = -3x^3 - 2x$ at $(-1, 5)$.

25. a. Determine the slope of the tangent to the parabola $y = 4x^2 + 5x - 2$ at the point whose x-coordinate is a.

b. At what point on the parabola is the tangent line parallel to the line $10x - 2y - 18 = 0$?

c. At what point on the parabola is the tangent line perpendicular to the line $x - 35y + 7 = 0$?