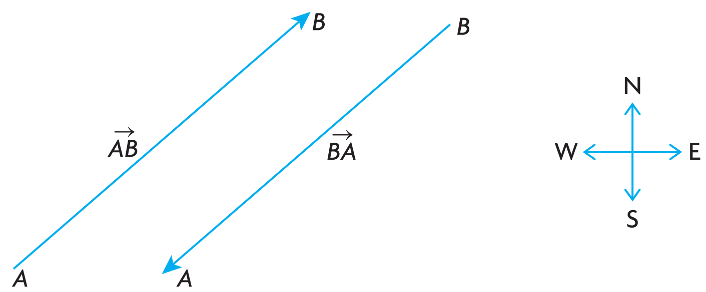
Section 6.1—An Introduction to Vectors

In mathematics and science, you often come in contact with different quantities. Some of these quantities, those whose **magnitude** (or size) can be completely specified by just one number, are called **scalars**. Some examples of scalars are age, volume, area, speed, mass, and temperature. On the other hand, some quantities (such as weight, velocity, or friction) require both a magnitude and a direction for a complete description and are called **vectors**.

Defining the Characteristics of Vectors

A vector can be represented by a directed line segment. A directed line segment has a length, called its magnitude, and a direction indicated by an arrowhead.

The diagram below can help to make the distinction between a vector and a scalar. If an airplane is travelling at a speed of 500 km/h, this description is useful, but for navigation and computational purposes, incomplete. If we add the fact that the airplane is travelling in a northeasterly direction, we now have a description of its velocity because we have specified both its speed and direction. This defines velocity as a vector quantity. If we refer to the speed of the airplane, we are describing it with just a single number, which defines speed as a scalar quantity.



Scale: 1 cm is equivalent to 100 km/h

In the diagram, $\overrightarrow{AB}$ is an example of a vector. In this case, it is a line segment running from A to B with its tail at A and head at B. Its actual size, or magnitude, is denoted by $|\overrightarrow{AB}|$. The magnitude of a vector is always non-negative. The vector $\overrightarrow{AB}$ could be used to represent the velocity of any airplane heading in a northeasterly direction at 500 km/h (using a scale of 1 cm to 100 km/h, i.e., $|\overrightarrow{AB}| = 5 \text{ cm} = 500 \text{ km/h}$). The direction of the "arrow" represents the direction of the airplane, and its length represents its speed.

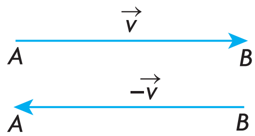
What is a Vector?

A vector is a mathematical quantity having both magnitude and direction.

In the diagram on the previous page, $\overrightarrow{BA}$ is a vector pointing from B to A. The vector $\overrightarrow{BA}$ represents an airplane travelling in a southwesterly direction at 500 km/h. Note that the magnitudes of the two vectors are equal, i.e., $|\overrightarrow{AB}| = |\overrightarrow{BA}|$, but that the vectors themselves are not equal because they point in opposite directions. For this reason, we describe these as **opposite vectors**.

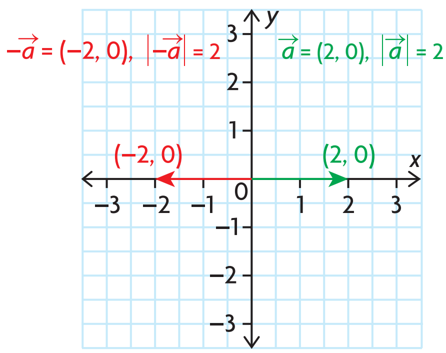
Opposite Vectors

Two vectors that are opposites have the same magnitude but point in opposite directions.



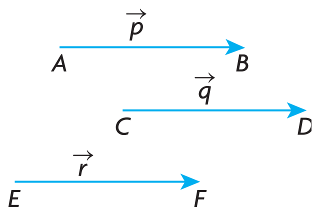
$\overrightarrow{AB}$ and $\overrightarrow{BA}$ are opposites, and $\overrightarrow{AB} = -\overrightarrow{BA}$. In this case, $|\overrightarrow{AB}| = |\overrightarrow{BA}|$ and the vectors are parallel but point in opposite directions. Vectors can also be represented with lower-case letters. In the diagram above, vectors $\vec{v}$ and $-\vec{v}$ have the same magnitude, i.e., $|\vec{v}| = |-\vec{v}|$, but point in opposite directions, so $\vec{v}$ and $-\vec{v}$ are also opposites.

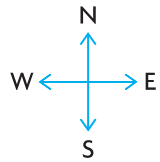
No mention has yet been made of using coordinate systems to represent vectors. In the diagram below, it is helpful to note that $\vec{a} = (2, 0)$ is a vector having its tail at the origin and head at $(2, 0)$; this vector has magnitude 2, i.e., $|\vec{a}| = 2$. Also, observe that $-\vec{a} = (-2, 0)$ and $|-\vec{a}| = 2$. The vectors $\vec{a}$ and $-\vec{a}$ are opposites.



It is not always appropriate or necessary to describe a quantity by both a magnitude and direction. For example, the description of the area of a square or rectangle does not require a direction. In referring to a person's age, it is clear what is meant by just the number. By their nature, quantities of this type do not have a direction associated with them and, thus, are not vectors.

Vectors are equal, or equivalent, if they have the same direction and the same magnitude. This means that the velocity vector for an airplane travelling in an easterly direction at 400 km/h could be represented by any of the three vectors in the following diagram.



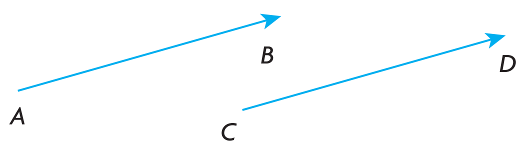


Scale: 1 cm is equivalent to 100 km/h

Notice that any one of these vectors could be translated to be **coincident** with either of the other two. (When vectors are translated, it means they are picked up and moved without changing either their direction or size.) This implies that the velocity vector of an airplane travelling at 400 km/h in an easterly direction from Calgary is identical to that of an airplane travelling at 400 km/h in an easterly direction from Toronto.

Note that, in the diagram above, we have also used lower-case letters to represent the three vectors. It is convenient to write the vector $\overrightarrow{AB}$ as $\vec{p}$, for example, and in this case $\vec{p} = \vec{q} = \vec{r}$.

Equal Vectors



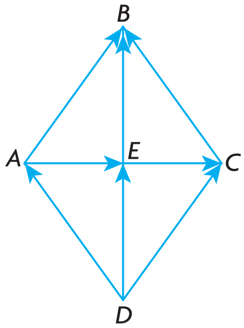
Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal (or equivalent) if and only if

1. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel to each other, and the direction from A to B is the same as the direction from C to D
2. the magnitude of $\overrightarrow{AB}$ equals the magnitude of $\overrightarrow{CD}$, i.e., $|\overrightarrow{AB}| = |\overrightarrow{CD}|$

EXAMPLE 1 Connecting vectors to two-dimensional figures

Rhombus $ABCD$ is drawn and its two diagonals AC and BD are drawn as shown. Name vectors equal to each of the following.

a. $\overrightarrow{AB}$ b. $\overrightarrow{DA}$ c. $\overrightarrow{EB}$ d. $\overrightarrow{AE}$



Solution

A rhombus is a parallelogram with its opposite sides parallel and the four sides equal in length. Thus, $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{DA} = \overrightarrow{CB}$ and $|\overrightarrow{AB}| = |\overrightarrow{DC}| = |\overrightarrow{DA}| = |\overrightarrow{CB}|$. Note that $\overrightarrow{AB} \neq \overrightarrow{DA}$ because these vectors have different directions, even though they have equal magnitudes, i.e., $|\overrightarrow{AB}| = |\overrightarrow{DA}|$.

Since the diagonals in a rhombus bisect each other, $\overrightarrow{AE} = \overrightarrow{EC}$ and $\overrightarrow{EB} = \overrightarrow{DE}$. Note also that, if the arrow had been drawn from C to D instead of from D to C, the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ would be opposites and would not be equal, even though they are of the same length. If these vectors are opposites, then the relationship between them can be expressed as $\overrightarrow{AB} = -\overrightarrow{CD}$. This implies that these vectors have the same magnitude but opposite directions.

In summary: a. $\overrightarrow{AB} = \overrightarrow{DC}$ b. $\overrightarrow{DA} = \overrightarrow{CB}$ c. $\overrightarrow{EB} = \overrightarrow{DE}$ d. $\overrightarrow{AE} = \overrightarrow{EC}$

In our discussion of vectors thus far, we have illustrated our ideas with **geometric vectors**. Geometric vectors are those that are considered without reference to coordinate axes. The ability to use vectors in applications usually requires us to place them on a coordinate plane. These are referred to as **algebraic vectors**; they will be introduced in the exercises and examined in detail in Section 6.5. Algebraic vectors will become increasingly important in our work.

IN SUMMARY

Key Ideas

- A vector is a mathematical quantity having both magnitude and direction, for example velocity.
- A scalar is a mathematical quantity having only magnitude, for example, speed.

Need to Know

- $\overrightarrow{AB}$ represents a vector running from A to B, with its tail at A and head at B.
- $|\overrightarrow{AB}|$ represents the magnitude of a vector and is always non-negative.
- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are opposite if they are parallel and have the same magnitude but opposite directions. It follows that $|\overrightarrow{AB}| = |\overrightarrow{BA}|$ and $\overrightarrow{AB} = -\overrightarrow{BA}$.
- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal if they are parallel and have the same magnitude and the same direction. It follows that $|\overrightarrow{AB}| = |\overrightarrow{CD}|$ and $\overrightarrow{AB} = \overrightarrow{CD}$.

Exercise 6.1

PART A

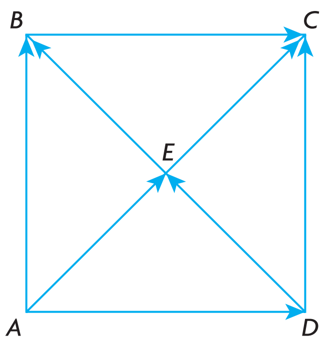
1. State whether each statement is true or false. Justify your decision.
   a. If two vectors have the same magnitude, then they are equal.
   b. If two vectors are equal, then they have the same magnitude.
   c. If two vectors are parallel, then they are either equal or opposite vectors.
   d. If two vectors have the same magnitude, then they are either equal or opposite vectors.

2. For each of the following, state whether the quantity is a scalar or a vector and give a brief explanation why: height, temperature, weight, mass, area, volume, distance, displacement, speed, force, and velocity.

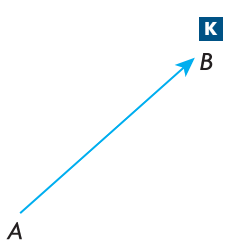
3. Friction is considered to be a vector because friction can be described as the force of resistance between two surfaces in contact. Give two examples of friction from everyday life, and explain why they can be described as vectors.

PART B

4. Square *ABCD* is drawn as shown below with the diagonals intersecting at *E*.



   a. State four pairs of equivalent vectors.
   b. State four pairs of opposite vectors.
   c. State two pairs of vectors whose magnitudes are equal but whose directions are perpendicular to each other.



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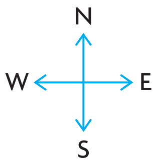
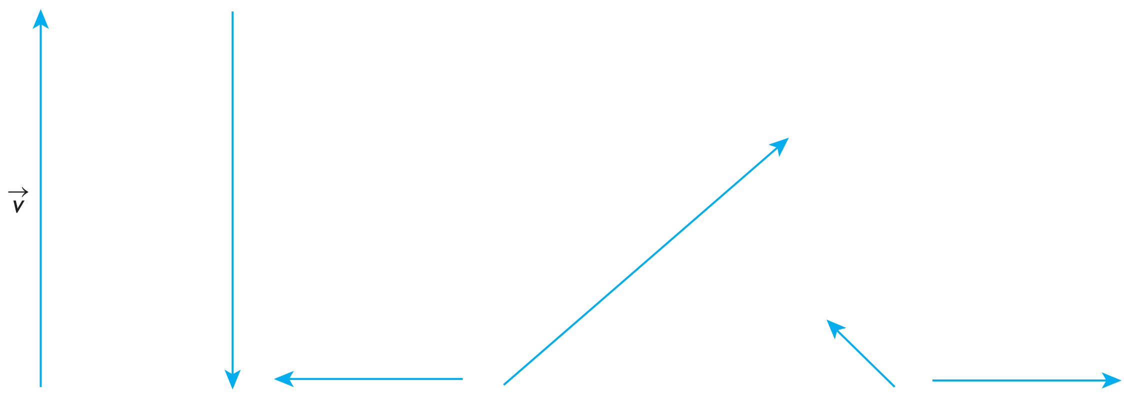
5. Given the vector $\overrightarrow{AB}$ as shown, draw a vector
   a. equal to $\overrightarrow{AB}$
   b. opposite to $\overrightarrow{AB}$
   c. whose magnitude equals $|\overrightarrow{AB}|$ but is not equal to $\overrightarrow{AB}$
   d. whose magnitude is twice that of $\overrightarrow{AB}$ and in the same direction
   e. whose magnitude is half that of $\overrightarrow{AB}$ and in the opposite direction



6. Using a scale of 1 cm to represent 10 km/h, draw a velocity vector to represent each of the following:
   a. a bicyclist heading due north at 40 km/h
   b. a car heading in a southwesterly direction at 60 km/h
   c. a car travelling in a northeasterly direction at 100 km/h
   d. a boy running in a northwesterly direction at 30 km/h
   e. a girl running around a circular track travelling at 15 km/h heading due east

7. The vector shown, $\vec{v}$, represents the velocity of a car heading due north at 100 km/h. Give possible interpretations for each of the other vectors shown.



a. b. c. d. e.

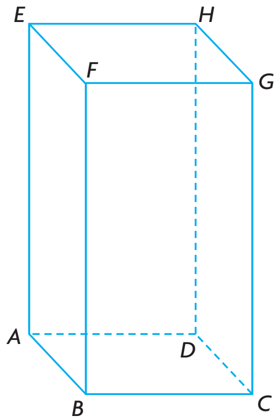
C

8. For each of the following vectors, describe the opposite vector.
   a. an airplane flies due north at 400 km/h
   b. a car travels in a northeasterly direction at 70 km/h
   c. a bicyclist pedals in a northwesterly direction at 30 km/h
   d. a boat travels due west at 25 km/h

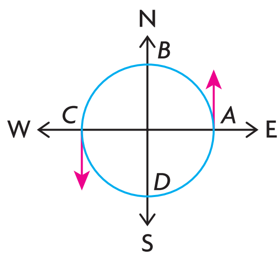
T 9. a. Given the square-based prism shown where $AB = 3$ cm and $AE = 8$ cm, state whether each statement is true or false. Explain.

i) $\overrightarrow{AB} = \overrightarrow{GH}$ ii) $|\overrightarrow{EA}| = |\overrightarrow{CG}|$ iii) $|\overrightarrow{AD}| = |\overrightarrow{DC}|$ iv) $\overrightarrow{AH} = \overrightarrow{BG}$

b. Calculate the magnitude of $\overrightarrow{BD}$, $\overrightarrow{BE}$, and $\overrightarrow{BH}$.



A 10. James is running around a circular track with a circumference of 1 km at a constant speed of 15 km/h. His velocity vector is represented by a vector tangent to the circle. Velocity vectors are drawn at points A and C as shown. As James changes his position on the track, his velocity vector changes.



a. Explain why James's velocity can be represented by a vector tangent to the circle.

b. What does the length of the vector represent?

c. As he completes a lap running at a constant speed, explain why James's velocity is different at every point on the circle.

d. Determine the point on the circle where James is heading due south.

e. In running his first lap, there is a point at which James is travelling in a northeasterly direction. If he starts at point A how long would it have taken him to get to this point?

f. At the point he has travelled $\frac{3}{8}$ of a lap, in what direction would James be heading? Assume he starts at point A.

PART C

11. $\overrightarrow{AB}$ is a vector whose tail is at $(-4, 2)$ and whose head is at $(-1, 3)$.

a. Calculate the magnitude of $\overrightarrow{AB}$.

b. Determine the coordinates of point D on vector $\overrightarrow{CD}$, if $C(-6, 0)$ and $\overrightarrow{CD} = \overrightarrow{AB}$.

c. Determine the coordinates of point E on vector $\overrightarrow{EF}$, if $F(3, -2)$ and $\overrightarrow{EF} = \overrightarrow{AB}$.

d. Determine the coordinates of point G on vector $\overrightarrow{GH}$, if $G(3, 1)$ and $\overrightarrow{GH} = -\overrightarrow{AB}$.